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THE JOURNAL

OF THE

Indian Mathematical Society

Vol. XII.] DECEMBER 1920. [No. 6

AN ALGEBRA OF ARITHMETICAL FUNCTIONS.

BY F. HALLBERG.

(*Continued from page* 169).

Algebraical Analysis.

§ 8. *The binomial theorem.*

Consider the following formal development.

$$\Big\{1 + F(s)\Big\}^k = 1 + \frac{k}{1} F(s) + \frac{k(k-1)}{2!} F^2(s) + \ldots$$

$$= \frac{1}{1^s} \cdot \{1 + f(1)\}^k + \frac{1}{2^s} \cdot \frac{k}{1!} \{1+f(1)\}^{k-1} f(2)$$

$$+ \frac{1}{3^s} \cdot \frac{k}{1!} \{1 + f(1)\}^{k-1} f(3)$$

$$+ \frac{1}{4^s} \cdot \left\{ \frac{k}{1!} \{1+f(1)\}^{k-1} f(4) + \frac{k(k-1)}{2!} \{1 + f(1)\}^{k-2} f^2(2) \right\}$$

$$+ \frac{1}{5^s} \cdot \frac{k}{1!} \{1 + f(1)\}^{k-1} f(5)$$

$$+ \frac{1}{6^s} \cdot \left\{ \frac{k}{1!} \{1 + f(1)\}^{k-1} f(6) \right.$$

$$\left. + \frac{k(k-1)}{1!\,1!} \{1 + f(1)\}^{k-2} f(2) f(3) \right\} + \ldots$$

Changing $[f(n)]$ to $[f(n)] - 1$, we obtain

$$F^k(s) = \frac{1}{1^s} \cdot f^k(1) + \frac{1}{2^s} \cdot \frac{k}{1!} f^{k-1}(1) f(2) + \frac{1}{3^s} \cdot \frac{k}{1!} f^{k-1}(1) f(3)$$

$$+ \frac{1}{4^s} \left\{ \frac{k}{1!} f^{k-1}(1) f(4) + \frac{k(k-1)}{2!} f^{k-2}(1) f^2(2) \right\}$$

$$+ \frac{1}{5^s} \cdot \frac{k}{1!} f^{k-1}(1) f(5) +$$

$$+ \frac{1}{6^s} \cdot \left\{ \frac{k}{1!} f^{k-1}(1) f(6) + \frac{k(k-1)}{1!\,1!} f^{k-2}(1) f(2) f(3) \right\} + \ldots$$

Consequently, using the general definition of a function of an **A. F.**, we deduce

$$[f(n)]^k = \Big[f^k(1), \frac{k}{1!} f^{k-1}(1) f(2), \frac{k}{1!} f^{k-1}(1) f(3),$$

$$\left\{ \frac{k}{1!} f^{k-1}(1) f(4) + \frac{k(k-1)}{2!} f^{k-2}(1) f^2(2) \right\}, \frac{k}{1!} f^{k-1}(1) f(5),$$

$$\left\{ \frac{k}{1!} f^{k-1}(1) f(6) + \frac{k(k-1)}{1!\,1!} f^{k-2}(1) f(2) f(3) \right\}, \ldots \Big]$$

$$= \Big[\sum_n \frac{k(k-1)\ldots(k-\lambda+1)}{\alpha_1!\,\alpha_2!\ldots\alpha_r!} f^{k-\lambda}(1) f^{\alpha_1}(a_1) f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r) \Big] \quad \ldots (8.1)$$

where the sum is extended to every decomposition of n in the form

$$n = a_1 a_2 \ldots a_r, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.11)$$

$a_1, a_2, \ldots, a_r$ being all different, λ being given by the formula

$$\lambda = \alpha_1 + \alpha_2 + \ldots + \alpha_r. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.12)$$

Since no assumption was made about the constant k, the formula (8.1) is perfectly general.

If k is a positive integer, the terms under the sign $\sum_n$, for which $\lambda > k$, vanish. In particular the case $k = 2$ gives

$$[f(n)]^2 = \Big[\sum_{d:n} f(d) \cdot f\left(\frac{n}{d}\right) \Big], \quad \ldots \quad \ldots \quad (8.2)$$

in agreement with (2.3).

In case the condition (5.1) *is satisfied*, we have, putting

$$[1]^k = [K(k, n)] = \Big[\sum_n \frac{k(k-1)\ldots(k-\lambda+1)}{\alpha_1!\,\alpha_2!\ldots\alpha_r!} \Big]$$

$$= [1, k, k, \tfrac{1}{2} k(k+1), k, k^2, k, \tfrac{1}{6} k(k+1)(k+2), \ldots] \ldots \quad (8.3)$$

as equivalent to (8.1), the formula

$$[f(n)]^k = [K(k, n) \cdot f(n)], \quad \ldots \quad \ldots \quad \ldots \quad (8.4)$$

which, for $k = 2$, gives, by (7.1)

$$[f(n)]^2 = [t(n) \cdot f(n)], \quad \ldots \quad \ldots \quad \ldots \quad (8.41)$$

while, for $k = -1$, (5.3) is obtained. The latter fact enables us to give a definite expression for the A. F. $[\mu(n)]$. The result is

$$\mu(n) = \sum_{n} \frac{(-1^{\lambda})\,\lambda\,!}{\alpha_1!\,\alpha_2!\,\ldots\,\alpha_r!}, \qquad \ldots \qquad \ldots \qquad (8.5)$$

(for $n > 1$).

It follows from our definitions, that the law of indices

$$[f(n)]^k \cdot [f(n)]^l = [f(n)]^{k+l} \qquad \ldots \qquad \ldots \qquad (8.6)$$

is valid. The elemental theorem of order $n = 6$, for instance, obtained from this formula by the aid of (8.1), has the appearance

$$f^k(1).\left\{ \frac{l}{1!} f^{l-1}(1) f(6) + \frac{l(l-1)}{1!\;1!} f^{l-2}(1)\, f(2)\, f(3) \right\}$$

$$+ \frac{k}{1!} f^{k-1}(1)\, f(2) . \frac{l}{1!} f^{l-1}(1)\, f(3) + \frac{k}{1!} f^{k-1}(1)\, f(3) . \frac{l}{1!} f^{l-1}(1)\, f(2)$$

$$+ \left\{ \frac{k}{1!} f^{k-1}(1) f(6) + \frac{k(k-1)}{1!\;1!} f^{k-2}(1) f(2)\, f(3) \right\} . f^l(1)$$

$$= \frac{k+l}{1!} f^{k+l-1}(1)\, f(6) + \frac{(k+l)(k+l-1)}{1!\qquad 1!} f^{k+l-2}(1)\, f(2)\, f(3),$$

a relation which is easily seen to be an identity.

Similarly we have

$$\left\{ [f(n)]^k \right\}^{\frac{1}{k}} = \left\{ [f(n)]^{\frac{1}{k}} \right\}^k = [f(n)] \;\ldots\ldots\; (8.7)$$

The first member of this equation gives, for instance, for $n = 6$:

$$\frac{1}{k} \cdot \frac{1}{1!} \left\{ f^k(1) \right\}^{\frac{1}{k}-1} . \left\{ \frac{k}{1!} f^{k-1}(1)\, f(6) + \right.$$

$$\left. + \frac{k(k-1)}{1!\;1!} f^{k-2}(1)\, f(2)\;\; f(3) \right\}$$

$$+ \frac{1}{k}\left(\frac{1}{k} - 1 \right) \cdot \frac{1}{1!\;1!} \left\{ f^k(1) \right\}^{\frac{1}{k}-2} .$$

$$. \left\{ \frac{k}{1!} f^{k-1}(1)\, f(2) . \frac{k}{1!} f^{k-1}(1)\, f(3) \right\} = f(6),$$

an obvious identity.

It follows from (8·3) that

$$[1] . [K(k, n)] = [K(k+1, n)] \;\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (8.8)$$

and hence, from a theorem of *Laguerre* (See I. S. and A. F., p. 255), by the aid of (6.1), (6.11), that

$$\sum_{n=1}^{\infty} K(k, n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n K(k+1, n) \quad \dots \quad \dots \quad (8.81)$$

Differentiation of this series gives

$$\sum_{n=1}^{\infty} K(k, n) \frac{n x^n}{(1-x^n)^2} = \sum_{n=1}^{\infty} n x^n K(k+1, n), \quad \dots \quad \dots \quad (8.82)$$

whilst by integration we obtain

$$\prod_{n=1}^{\infty} (1-x^n)^{\frac{K(k, n)}{n}} = \text{Exp} \left\{ -\sum_{n=1}^{\infty} \frac{x_n}{n} K(k+1, n) \right\}. \quad \dots \quad 8.83)$$

It would be of the greatest importance for the Theory of Numbers, if the order of

$$\sum_{r=1}^{n} K(k, r)$$

could be obtained, when $n \to \infty$. This would, in particular, give us the order of $\sigma(n)$, defined by (6.5), and thus solve the difficulty of Riemann's proposition over the imaginary roots of the ζ-function. (See *Encycl.*, l.c., p. 351; or I. S. & A. F., p. 263, where this question is shortly discussed.) It should be noticed, that $K(k, n)$, and thus also the required sum, may be regarded as a continuous function of k, a fact which may be of use, if the solution of the above problem is attempted. The algebraical difficulties, however, are certainly very great.

As a verification of some of the above methods, consider the case $k = -1$.

We have

$$\left[f(n) \right]^{-1} = \frac{1}{1 - \{1 - [f(n)]\}}$$

$$= 1 + \{1 - [f(n)]\} + \{1 - [f(n)]\}^2$$

$$+ \{1 - [f(n)]\}^3 + \dots$$

If we now develop the brackets of this series by the aid of (8.1) and then add the series of A. F. obtained, the result is

$$\left[f(n)\right]^{-1} = \left[f^{-1}(1), \frac{-1}{1!} f^{-2}(1)\, f(2), \frac{-1}{1!} f^{-2}(1)\, f(3),\right.$$

$$\left.\left\{\frac{-1}{1!} f^{-2}(1)\, f(4) + \frac{(-1)(-2)}{2!} f^{-3}(1)\ f^{2}(2)\right\}, \ldots\right] \quad \ldots \quad (8.9)$$

in agreement with (8.1), provided $f(1) \pm 0$. Hence, if this condition is satisfied, division by $[f(n)]$ is always possible.

§ 9. *Maclaurin's theorem.*

We have, γ being an analytic function (subject to certain conditions)

$$\gamma(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!} \gamma^{(r)}(0). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9.1)$$

In this formula, put $[\, f(n)\,]$ for z.

$$\therefore \quad \gamma[f(n)] = [g(n)] = \sum_{r=0}^{\infty} \frac{[f(n)]^r}{r!} \gamma^{(r)}(0). \quad \ldots \quad \ldots \quad (9.2)$$

If we now develop the powers of $[\, f(n)\,]$, occurring in the terms of the series, by (8·1), and afterwards reduce, we obtain the value of $[\, g(n)]$. The result is

$$[g(n)] = \left[\sum_{r=0}^{\infty} \frac{f^r(1)}{r!} \gamma^{(r)}(0), f(2) \sum_{r=0}^{\infty} \frac{r+1}{1!} \frac{f^r(1)}{(r+1)!} \gamma^{(r+1)}(0),\right.$$

$$f(3) \sum_{r=0}^{\infty} \frac{r+1}{1!} \frac{f^r(1)}{(r+1)!} \gamma^{(r+1)}(0), \left\{ f(4) \sum_{r=0}^{\infty} \frac{r+1}{1!} \frac{f^r(1)}{(r+1)!} \gamma^{(r+1)}(0)\right.$$

$$\left.\left. + f^2(2) \sum_{r=0}^{\infty} \frac{(r+1)(r+2)}{2!} \frac{f^r(1)}{(r+2)!} \gamma^{(r+2)}(0) \right\}, \ldots \right], \quad \ldots \quad (9.21)$$

which may be written

$$\gamma\left[f(n)\right] = \left[\gamma\{f(1)\}, \frac{f(2)}{1!} \gamma'\{f(1)\}, \frac{f(3)}{1!} \gamma'\{f(1)\},\right.$$

$$\left\{\frac{f(4)}{1!} \gamma'\{f(1)\} + \frac{f^2(2)}{2!} \gamma''\{f(1)\}\right\}, \frac{f(5)}{1!} \gamma'\{f(1)\},$$

$$\left.\left\{\frac{f(6)}{1!} \gamma'\{f(1)\} + \frac{f(2)\,.\,f(3)}{1!\,1!} \gamma''\{f(1)\}, \ldots\right]\right.$$

$$= \left[\sum_{n} \frac{f^{\alpha_1}(a_1)\ f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\ \alpha_2! \ \ldots \ \alpha_r!} \gamma^{(\lambda)}\{f(1)\}\right], \quad \ldots \quad (9.3)$$

the notation being explained in (8.11), (8.12).

We see at once, that formula (8 . 1) is a special case of (9 . 3), obtained for $\gamma(z) = z^k$. The reason, while the binominal theorem is considered before the general case, is that in Maclaurin's theorem powers of the A. F. employed occur, for the development of which the former theorem is necessary. It should be noticed, however, that Maclaurin's theorem could also have been obtained directly from the fundamental definition (by the aid of Dirichlet's series, as in § 8, or otherwise). The above method has been chosen, because it makes the reciprocal nature of the two theorems apparent.

Assume the condition (5 . 1) to be satisfied. The formula (9 . 3) is then reduced to the following :—

$$\gamma\,[f(n)] = [f(n)\,.\,\eta(n)], \qquad \ldots \qquad \ldots \qquad (9\,.\,4)$$

where

$$[\eta(n)] = \gamma\,[1], \qquad \ldots \qquad \ldots \qquad \ldots \quad (9\,.\,41)$$

and thus

$$\eta(n) = \sum_{n} \frac{\gamma^{(\lambda)}(1)}{\alpha_1!\,\alpha_2!\,\ldots\,\alpha_r!}$$

$$= \sum_{r=1}^{\infty} \frac{K(r, n)}{r!}\,\gamma^{(r)}(0), \qquad \ldots \qquad \ldots \quad (9\,.\,411)$$

for $n > 1$, $K(k, n)$ being the A. F. defined by (8 . 3). The last expression is obtained directly from (9 . 2) by putting $[f(n)] = [1]$, which method gives

$$\gamma\,[1] = \gamma(0) + \sum_{r=1}^{\infty} \frac{[K(r, n)]}{r!}\,\gamma^{(r)}(0).$$

The identity between the two expressions for $\eta(n)$ may readily be established in each special case. For instance, $n = 6$ gives

$$\sum_{r=1}^{\infty} \frac{K(r, 6)}{r!}\,\gamma^{(r)}(0) = \sum_{r=1}^{\infty} \frac{r^2}{r!}\,\gamma^{(r)}(0)$$

$$= \sum_{r=1}^{\infty} \frac{1}{(r-1)!}\,\gamma^{(r)}(0) + \sum_{r=1}^{\infty} \frac{1}{(r-2)!}\,\gamma^{(r)}(0)$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!}\,\gamma^{(r+1)}(0) + \sum_{r=0}^{\infty} \frac{1}{r!}\,\gamma^{(r+2)}(0)$$

$$= \frac{1}{1!}\gamma'(1) + \frac{1}{1!\,1!}\gamma''(1) = \eta(6).$$

As an example of (9 . 4), let $\gamma(z) = (1 + z)^k$, and consider $z = [1]$.

$$\therefore\quad (1 + [1])^k = [2, 1, 1, 1, \ldots]^k$$

$$= \Big[2^k,\ \frac{k}{1!}\,2^{k-1},\ \frac{k}{1!}\,2^{k-1},\ \Big\{\frac{k}{1!}\,2^{k-1} + \frac{k(k-1)}{2!}\,2^{k-2}\Big\},$$

$$\frac{k}{1!}\,2^{k-1},\ \Big\{\frac{k}{1!}\,2^{k-1} + \frac{k}{1!\,1!}\,2^{k-2}\Big\}, \ldots\Big]$$

$$= \Big[2^k,\ \sum_{r=1}^{\infty} r\ C_r^k,\ \sum_{r=1}^{\infty} r\ C_r^k,\ \sum_{r=1}^{\infty}\frac{r(r+1)}{2}\,C_r^k,$$

$$\sum_{r=1}^{\infty} r\ C_r^k,\ \sum_{r=1}^{\infty} r^2\,C_r^k, \ldots\Big],$$

the nth element $(n > 1)$ being

$$\sum_{r=1}^{\infty} K(r, n)\,C_r^k.$$

§ 10. *Tho exponential and logarithmic functions.*

Put in (9 . 3) $\gamma(z) = e^z$.

We find

$$e^{[f(n)]} = e^{f(1)}\Big[1,\ \frac{f(2)}{1!},\ \frac{f(3)}{1!},\ \Big\{\frac{f(4)}{1!} + \frac{f^2(2)}{2!}\Big\},$$

$$\frac{f(5)}{1!},\ \Big\{\frac{f(6)}{1!} + \frac{f(2)\,.\,f(3)}{1!\,1!}\Big\}, \ldots\Big]$$

$$= e^{f(1)}\Big[\sum_n \frac{f^{\alpha_1}(a_1)}{\alpha_1!}\,\frac{f^{\alpha_2}(a_2)\ \ldots\ f^{\alpha_r}(a_r)}{\alpha_2!\ \ldots\ \alpha_r!}\Big], \quad \ldots (10 . 1,$$

which, *in case* $f(n)$ *satisfies the relation* (*5 . 1*), becomes

$$e^{[f(n)]} = e^{[f(n)\,.\,\varepsilon(n)]}, \quad \ldots \quad \ldots \quad \ldots (10 . 2)$$

where

$$[\varepsilon(n)] = e^{[1]-1}; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (10.21)$$

$$\varepsilon(1) = 1; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (10.210)$$

$$\varepsilon(n) = \sum_n \frac{1}{\alpha_1!\,\alpha_2!\ldots\alpha_r!} =$$

$$= \frac{1}{e}\sum_{r=1} \frac{K(r, n)}{r!}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10.211)$$

for $n > 1$

In a similar manner we treat the logarithmic function

$$\gamma(z) = \log z.$$

We obtain

$$\log[f(n)] = \Big[\log f(1), \frac{1}{1!}\frac{f(2)}{f(1)}, \frac{1}{1!}\frac{f(3)}{f(1)},$$

$$\left\{\frac{1}{1!}\frac{f(4)}{f(1)} - \frac{1!}{2!}\left\{\frac{f(2)}{f(1)}\right\}^2\right\}, \frac{1}{1!}\frac{f(5)}{f(1)},$$

$$\left\{\frac{1}{1!}\frac{f(6)}{f(1)} - \frac{1!}{1!\,1!}\frac{f(2)}{f(1)}, \frac{f(3)}{f(1)}\right\}, \ldots\Big] \quad \ldots \quad (10.3)$$

the nth element for $n > 1$ of the A. F. being

$$\sum_n \frac{f^{\alpha_1}(a_1) f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\ \alpha_2!\ \ldots\ \alpha_r!} \quad \frac{(-1)^{\lambda-1}.(\lambda-1)!}{f^{\lambda}(1)}. \quad \ldots \quad (10.31)$$

In case (5·1) is satisfied, this reduces to

$$\log[f(n)] = [f(n) \,.\, l(n)], \quad \ldots \quad \ldots \quad (10.4)$$

where

$$[l(n)] = \log[1]; \quad \ldots \quad \ldots \quad \ldots \quad (10.41)$$

$$l(1) = 0; \quad \ldots \quad \ldots \quad \ldots \quad (10.410)$$

$$l(n) = \sum_n \frac{(-1)^{\lambda-1}(\gamma-1)!}{\alpha_1!\,\alpha_2!\ldots\alpha_r!}, \ldots \quad \ldots \quad (10.411)$$

for $n > 1$, the second expression for $l(n)$ being in this case unvalid.

The A. F. defined by (10.1), (10.3) now obey all the laws of the ordinary exponential and logarithmic functions. We have, for instance,

$$e^{[f(n)]} \,.\, e^{[g(n)]}, = e^{[f(n)+g(n)]}; \quad \ldots \quad \ldots \quad \ldots \quad (10.5)$$

$$\log[f(n)] + \log[g(n)] = \log\ \{[f(n)] \,.\, [g(n)]\}; \quad \ldots \quad (10.6)$$

$$\log e^{[f(n)]} = e^{\log[f(n)]} = [f(n)] \quad \ldots \quad \ldots \quad \ldots \quad (10.7)$$

As an example, it will be sufficient to consider the first part of (10.7). We get

$$\log e^{[f(n)]} = \log e^{f(1)} + \log\left[\sum_n \frac{f^{\alpha_1}(a_1)\ f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\ \alpha_2!\ \ldots\ \alpha_r!}\right]$$

$$= f(1) + \Big[\log 1,\ \frac{1}{1!}\frac{f(2)}{1!},\ \frac{1}{1!}\frac{f(3)}{1!},\ \Big\{\frac{1}{1!}\frac{f(4)}{1!} + \frac{1}{1!}\frac{f^2(2)}{2!}$$

$$-\frac{1!}{2!}\Big\{\frac{f(2)}{1!}\Big\}^2\Big\},\ \frac{1}{1!}\frac{f(5)}{1!},\ \Big\{\frac{1}{1!}\frac{f(6)}{1!} + \frac{1}{1!}\frac{f(2).f(3)}{1!\ \ 1!}$$

$$-\frac{1!}{1!\,1!}\frac{f(2)}{1!}\frac{f(3)}{1!}\Big\},\ \ldots\Big] = [f(1), f(2), f(3), f(4), f(5), f(6), \ldots]$$

$$= [f(n)].$$

§ 11. *Extended definition of a power.*

As yet we have only dealt with powers of the form

$$[f(n)]^k,\ e^{[f(n)]}.$$

We are now in a position to extend our definitions to the most general case. We define

$$[f(n)]^{[g(n)]} = e^{[g(n)].\log[f(n)]}. \qquad \ldots \qquad \ldots \qquad \ldots \quad (11.1)$$

Now, by (1.03)

$$[g(n)].\log[f(n)] = \Big[g(1)\log f(1),\ \Big\{g(2)\log f(1) +$$

$$\frac{g(1)}{1!}\frac{f(2)}{f(1)}\Big\},\ \Big\{g(3)\log f(1) + \frac{g(1)}{1!}\frac{f(3)}{f(1)}\Big\},\ \Big\{g(4)\log f(1) +$$

$$\frac{g(2)}{1!}\frac{f(2)}{f(1)} + g(1)\Big\{\frac{1}{1!}\frac{f(4)}{f(1)} - \frac{1!}{2!}\Big(\frac{f(2)}{f(1)}\Big)^2\Big\}\Big\},\ \ldots\Big].$$

Hence, by (10.1)

$$[f(n)]^{[g(n)]} = f(1)\,e^{g(1)}\Big[1, \frac{1}{1!}\Big\{g(2)\log f(1) + \frac{g(1)}{1!}\frac{f(2)}{f(1)}\Big\},$$

$$\frac{1}{1!}\Big\{g(3)\log f(1) + \frac{g(1)}{1!}\frac{f(3)}{f(1)}\Big\},\ \Big\{\frac{1}{1!}\Big\{g(4)\log f(1) + \frac{g(2)}{1!}\frac{f(2)}{f(1)}$$

$$+ g(1)\Big\{\frac{1}{1!}\frac{f(4)}{f(1)} - \frac{1!}{2!}\Big(\frac{f(2)}{f(1)}\Big)^2\Big\}\Big\} + \frac{1}{2!}\Big\{g(2)\log f(1) +$$

$$+ \frac{g(1)}{1!}\frac{f(2)}{f(1)}\Big\}^2\Big\},\ \ldots\Big] \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (11\cdot2)$$

§ 12. *The trigonometrical functions.*

If in (9.3) we assume

$$\gamma(z) = \sin z,$$

we obtain

$$\sin[f(n)] = \Big[\sin f(1),\ \frac{f(2)}{1!}\cos f(1),\ \frac{f(3)}{1!}\cos f(1),$$

$$\left\{\frac{f(4)}{1!}\cos f(1) - \frac{f^2(2)}{2!}\sin f(1)\right\},\ \frac{f(5)}{1!}\cos f(1),$$

$$\left\{\frac{f(6)}{1!}\cos f(1) - \frac{f(2)}{1!}\cdot\frac{f(3)}{1!}\sin f(1)\right\},\ \ldots\Big]$$

$$= \left[\sum_n \frac{f^{\alpha_1}(a_1)\, f^{\alpha_2}(a_2)\ \ldots\ f^{\alpha_r}(a_r)}{\alpha_1!\ \ \alpha_2!\ \ \ldots\ \ \alpha_r!}\sin\left\{f(1) + \lambda\frac{\pi}{2}\right\}\right]. \quad \ldots \quad (12.1)$$

Similarly for

$$\gamma(z) = \cos z$$

$$\cos[f(n)] = \Big[\cos f(1),\ \left\{-\frac{f(2)}{1!}\sin f(1)\right\},\ \left\{-\frac{f(3)}{1!}\sin f(1)\right\},$$

$$\left\{-\frac{f(4)}{1!}\sin f(1) - \frac{f^2(2)}{2!}\cos f(1)\right\},\ \left\{-\frac{f(5)}{1!}\sin f(1)\right\},$$

$$\left\{-\frac{f(6)}{1!}\sin f(1) - \frac{f(2)}{1!}\cdot\frac{f(3)}{1!}\cos f(1)\right\},\ \ldots\Big] =$$

$$= \left[\sum_n \frac{f^{\alpha_1}(a_1)\, f^{\alpha_2}(a_2)\ \ldots\ f^{\alpha_r}(a^r)}{\alpha_1!\ \ \alpha_2!\ \ \ldots\ \ \alpha_r!}\cos\left\{f(1) + \lambda\frac{\pi}{2}\right\}\right]. \quad (12.2)$$

These expressions evidently become much simplified, if $f(1)$ is a multiple of $\frac{\pi}{4}$.

If (5.1) *is satisfied*, (12.1) becomes

$$\sin[f(n)] = [f(n) \cdot s(n)], \quad \ldots \quad \ldots \quad (12.3)$$

where

$$[s(n)] = \sin[1]; \quad \ldots \quad \ldots \quad (12.31)$$

$$s(1) = \sin 1; \quad \ldots \quad \ldots \quad (12.310)$$

$$s(n) = \sum_n \frac{\sin\left(1 + \lambda\frac{\pi}{2}\right)}{\alpha_1!\,\alpha_2!\ldots\alpha_r!}$$

$$= \sum_{r=1}^{\infty}(-1)^{r-1}\frac{K(2r-1, n)}{(2r-1)!}, \quad \ldots \quad (12.311)$$

for $n > 1$.

Similarly, (5.1) *being satisfied*, we have

$$\cos[f(n)] = [f(n) \cdot c(n)], \quad \ldots \quad \ldots \quad (12.4)$$

where

$$[c(n)] = \cos[1]; \quad \ldots \quad \ldots \quad \ldots \quad (12.41)$$

$$c(1) = \cos 1; \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12.410)$$

$$c(n) = \sum_{n} \frac{\cos\left(1 + \lambda \frac{\pi}{2}\right)}{\alpha_1!\, \alpha_2! \ldots \alpha_r!}$$

$$= \sum_{r=1}^{\infty} (-1)^r \frac{K(2r, n)}{2r!}, \quad \ldots \quad = \quad (12.411)$$

for $n > 1$.

We may now define and obtain expressions for other trigonometrical functions as in ordinary trigonometry, and thus carry over the entire trigonometry to our extended algebraical domain.

We note in particular

$$\cos[f(n)] + i \sin[f(n)]$$

$$= \{\cos f(1) + i \sin f(1)\} \left[\sum_{n} \frac{f^{\alpha_1}(a_1)}{\alpha_1!} \frac{f^{\alpha_2}(a_2)}{\alpha_2!} \ldots \frac{f^{\alpha_r}(a_r)}{\alpha^r!} i^{\lambda}\right]$$

$$= e^{i\,[f(n)]}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12.5)$$

from which we deduce the equivalent of De Moivre's theorem

$$\{\cos[f(n)] + i \sin[f(n)]\}^{[g(n)]} =$$

$$= \cos[f(n)] \cdot [g(n)] + i \sin[f(n)] \cdot [g(n)], \quad \ldots \quad (12.6)$$

leaving out the discussion of multiple values. Interesting, but complicated, identities may be obtained by developing the left side of this formula by the aid of (11.2).

As an example of (12.6), let $[g(n)] = k$, $n = 6$

$$\therefore \frac{k}{1!} \{\cos f(1) + i \sin f(1)\}^{k-1} \left\{-\frac{f(6)}{1!} \sin f(1)\right.$$

$$- \frac{f(2) \cdot f(3)}{1!\ \ 1!} \cdot \cos f(1) + i \frac{f(6)}{1!} \cos f(1) - i \frac{f(2) \cdot f(3)}{1!\ \ 1!} \sin f(1) \left.\right\}$$

$$+ \frac{k(k-1)}{1!\,1!} \cdot \{\cos f(1) + i \sin f(1)\}^{k-2}$$

$$\cdot \left\{-\frac{f(2)}{1!} \sin f(1) + i \frac{f(2)}{1!} \cos f(1)\right\} \cdot \left\{-\frac{f(3)}{1!} \sin f(1) + \right.$$

$$\left. + i \frac{f(3)}{1} \cos f(1)\right\}$$

$$= -\frac{k\,f(6)}{1!}\sin\{k\,f(1)\} - \frac{k\,f(2)}{1!}\cdot\frac{k\,f(3)}{1!}\cos\{k f(1)\} +$$

$$+\, i\,\frac{k\,f(6)}{1!}\cos\{k\,f(1)\} - i\,\frac{k\,f(2)}{1!}\cdot\frac{k\,f(3)}{1!}\sin\{k\ f(1)\},$$

which relation is easily seen to be an identity.

§ 13. *The inverse circular functions.*

Let in (9 . 3)

$$\gamma(z) = \tan^{-1} z,$$

which assumption gives

$$\tan^{-1}[f(n)] = \Big[\tan^{-1}f(1),\ f(2)\,\frac{1}{1+f^2(1)},\ f(3)\,\frac{1}{1+f^2(1)},$$

$$\Big\{f(4)\,\frac{1}{1+f^2(1)} - f^2(2)\,\frac{f(1)}{[1+f^2(1)]^2}\Big\},\ f(5)\,\frac{1}{1+f^2(1)},$$

$$\Big\{f(6)\,\frac{1}{1+f^2(1)} - f(2)\,f(3)\,\frac{2f(1)}{[1+f^2(1)]^2}\Big\},\ \dots\Big]$$

$$= \Big[\sum_n \frac{f^{\alpha_1}(a_1)}{\alpha_1!}\,\frac{f^{\alpha_2}(a_2)}{\alpha_2!}\cdots\frac{f^{\alpha_r}(a_r)}{\alpha_r!}\,(-1)^{\lambda-1}(\lambda-1)!\sin\lambda\theta\,.\sin^{\lambda}\theta\Big], \quad \dots \quad \dots (13.1)$$

where

$$\theta = \frac{\pi}{2} - \tan^{-1} f(1), \quad \dots \quad \dots (13.11)$$

the first element being $\tan^{-1} f(1)$.

When the relation (5 . 1) *is satisfied,* (13 . 1) gives

$$\tan^{-1}[f(n)] = [f(n)\,.\,\tau(n)]$$

$$= \Big[\frac{\pi}{4},\ \frac{f(2)}{2},\ \frac{f(3)}{2},\ \frac{f(4)}{4},\ \frac{f(5)}{2},\ 0,\ \dots\Big] \quad \dots \quad \dots (13.2)$$

where

$$[\tau(n)] = \tan^{-1}[1]; \quad \dots \quad \dots (13.21)$$

$$\tau(1) = \tan^{-1} 1 = \frac{\pi}{4}; \quad \dots \quad \dots (13.210)$$

$$\tau(n) = \sum_n \frac{(-1)^{\lambda-1}\,2^{-\frac{\lambda}{2}}(\lambda-1)!\ \sin\lambda\frac{\pi}{4}}{\alpha_1!\ \alpha_2!\ \dots\ \alpha_r!} =$$

$$= \sum_{r=1}^{\infty}(-1)^{r-1}\frac{K(2r-1,\,n)}{2r-1}. \quad \dots \quad \dots (13.211)$$

Here it should be noticed, however, that the series in the last expression for $\tau(n)$ converges only for $n = 1$. For higher n-values we must use Borel's method of summing an oscillatory series. Thus we have, for instance, for $n = 4$

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{K(2r-1, 4)}{2r-1} = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{(2r-1) \cdot 2r}{2(2r-1)}$$

$$= \sum_{r=1}^{\infty} (-1)^{r-1} r = \tfrac{1}{4},$$

and for $n = 6$

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{K(2r-1, 6)}{2r-1} = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{(2r-1)^2}{2r-1}$$

$$= \sum_{r=1}^{\infty} (-1)^{r-1} (2r-1) = 0,$$

which is evidently correct.

Other inverse circular functions may be treated in the same manner and the theory of these functions may be built up exactly as in ordinary analysis.

§ 14. *The ζ-function.*

It is to be noticed, that the formula $(9 \cdot 3)$ is not a consequence of Maclaurin's theorem, but much more general. To realize this, it is necessary to consider some other devolopment of an analytic function. We choose the ζ-function as an example.

$$\zeta[f(n)] = \sum_{r=1}^{\infty} r^{-[f(n)]} = \sum_{r=1}^{\infty} e^{[-f(n) \log r]}$$

$$= \Big[\sum_{r=1}^{\infty} r^{-f(1)},\ -\frac{f(2)}{1!} \sum_{r=2}^{\infty} r^{-f(1)} \log r,\ -\frac{f(3)}{1!} \sum_{r=2}^{\infty} r^{-f(1)} \log r,$$

$$\Big\{-\frac{f(4)}{1!} \sum_{r=2}^{\infty} r^{-f(1)} \log r + \frac{f(2)}{2!} \sum_{r=2}^{\infty} r^{-f(1)} \log^2 r\Big\},\ \ldots\Big]$$

$$= \left[\zeta\{f(1)\}, \frac{f(2)}{1!}\zeta'\{f(1)\}, \frac{f(3)}{1!}\zeta'\{f(1)\}, \left\{\frac{f(4)}{1!}\zeta'\{f(1)\} + \frac{f^2(2)}{2!}\zeta''\{f(1)\}\right\}, \ldots \right]$$

$$= \left[\sum_{n} \frac{f^{\alpha_1}(a_1)\, f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\ \alpha_2!\ \ldots\ \alpha_r!} \zeta^{(\lambda)}\{f(1)\} \right], \qquad \ldots (14 . 1)$$

in agreement with formula (9 . 3).

In a similar manner we may treat a Dirichlet's series in general.

We may now extend our fundamental definition as follows:

If an arithmetical function $[f(n)]$ *and a function* $\gamma(z)$ *are given; then the arithmetical function* $\gamma[f(n)]$ *is defined by formula (9 . 3), provided* $\gamma(z)$ *is analytic for* $z = f(1)$.

The Infinitesimal Calculus.

§ 15. *The Differential Calculus.*

The process of differentiation of an A. F. may be arrived at in various ways.

Put in the formula (9.3) γ' instead of γ.

We obtain

$$\gamma'[f(n)] = \left[\sum_{n} \frac{f^{\alpha_1}(a_1)\ f^{\alpha_2}(a_2) \ldots f^{\alpha_r}(a_r)}{\alpha_1!\ \alpha_2!\ \ldots\ \alpha_r!}\ \gamma^{(\lambda+1)}\{f(1)\} \right]$$

$$= \left[\frac{d\ \{g\ (n)\}}{d\ \{f\ (1)\}}\right] = \frac{d\ \gamma\ [f\ (n)]}{d\ [f\ (n)]}, \quad \ldots \quad \ldots \quad \ldots \quad (15{\cdot}1)$$

by definition.

As examples, consider the particular functions, examined in detail above. We get

from (8.1): $\dfrac{d\ [f\ (n)]^k}{d\ [f\ (n)]} = k\ [f\ (n)]^{k-1}$; ... ... (15·11)

from (10·1): $\dfrac{d\ e^{[f\ (n)]}}{d\ [f\ (n)]} = e^{[f\ (n)]}$; ... ... (15·12)

from (10·3), and (8·9): $\dfrac{d \log\ [f\ (n)]}{d\ [f\ (n)]} = [f\ (n)]^{-1}$; ... ... (15·13)

from (12·1), and (12·2):

$\dfrac{d \sin\ [f\ (n)]}{d\ [f\ (n)]} = \cos\ [f\ (n)]$, ... (15·14)

$\dfrac{d \cos\ [f\ (n)]}{d\ [f\ (n)]} = -\sin\ [f\ (n)]$, ... (15·15)

etc.

It is not diffucult to see, that we may build up the whole theory, following the course generally adopted in the differential calculus.

We may also arrive at these results in the following manner. Assume that the elements of the A. F. considered are continuous functions of the variable x.

We define

$$\frac{d\,[f(x, n)]}{dx} = \underset{\Delta x = 0}{\mathrm{Lt}} \frac{[f(x + \Delta x, n)] - [f(x, n)]}{\Delta x}$$

$$= \underset{\Delta x = 0}{\mathrm{Lt}} \left[\frac{f(x + \Delta x, n) - f(x, n)}{\Delta x}\right] = \left[\frac{d f(x, n)}{dx}\right] . \quad \ldots (15.2)$$

Let further

$$\gamma\,[f(x, n)] = [g(x, n)] , \quad \ldots \quad \ldots \quad \ldots \quad (15.3)$$

and define

$$\frac{d\,[g(x, n)]}{d\,[f(x, n)]} = \gamma'\,[f(x, n)]$$

$$= \left[\frac{d\,g(x, n)}{dx}\right] \cdot \left[\frac{d f(x, n)}{dx}\right]^{-1} . \quad \ldots \quad (15.4)$$

In order to establish the identity between the two definitions, consider (9.3), where instead of $f(n)$, we put $f(x, n)$. If our proposition is correct, we must have identically (by (15.1), (15.4)).

$$\gamma'\,[f(x, n)] = [\gamma'\{f(x, 1)\}, \frac{f(x, 2)}{1!}\gamma''\{f(x, 1)\}, \frac{f(x, 3)}{1!} .$$
$$.\gamma''\{f(x, 1)\}, \left\{\frac{f(x, 4)}{1!}\gamma''\{f(x. 1) + \frac{f^2(x, 2)}{2!}\gamma'''\{f(x, 1)\}\right\}, \ldots]$$
$$= \Big[\gamma'\{f\ x, 1)\}.f'(x, 1), \left\{\frac{f(x, 2)}{1!}\gamma''\{f(x, 1)\}\,f'(x, 1) + \right.$$
$$\left.\frac{f'(x, 2)}{1!}\gamma'\{f(x, 1)\}\right\}, \left\{\frac{f(x, 3)}{1!}\gamma''\{f(x, 1)\}\,f'(x, 1) + \right.$$
$$\left.\frac{f'(x, 3)}{1!}\gamma'\{f(x, 1)\}\right\}, \left\{\frac{f(x, 4)}{1!}\gamma''\{f(x, 1)\}\,f'(x, 1) + \right.$$
$$\frac{f'(x, 4)}{1!}\gamma'\{f(x, 1)\} + \frac{f^2(x, 2)}{2!}\gamma'''\{f(x, 1)\}\,f'(x, 1) +$$
$$\left.\frac{2f(x, 2)\,f'(x. 2)}{2!}\gamma''\{f(x, 1)\}\right\}, \ldots\Big] \cdot \Big[\{f'(x, 1)\}^{-1},$$
$$\frac{-1}{1!}\{f'(x, 1)\}^{-2} f'(x, 2), \frac{-1}{1!}\{f'(x, 1)\}^{-2} f'(x, 3),$$
$$\left\{\frac{-1}{1!}\{f'(x, 1)\}^{-2} f'(x, 4) + \right.$$
$$\left.\frac{(-1)(-2)}{2!}\{f'(x, 1)\}^{-3}\{f'(x, 2)\}^2\right\}, \ldots\Big] \quad \ldots \quad (15.5)$$

This is evidently the case, as may be seen by actually carrying out the multiplication to the right of the sign of equality.

It is clear, that an A. F. may also be regarded as a constant during the process of differentiation. As an example, consider

$$\frac{d\,[f\,(n)]^k}{d\,k} = [f\,(n)]^k \,.\, \log\,[f\,(n)]$$

$$= \Big[\, f^k(1) \log f\,(1),\ \frac{f\,(2)}{1\,!} f^{k-1}(1)\ \{1 + k \log f\,(1)\},$$

$$\frac{f\,(3)}{1\,!} f^{k-1}(1)\ \{1 + k \log f(1)\},\ \Big\{\frac{f\,(4)}{1\,!} f^{k-1}(1)\,\{1 + k \log f\,(1)\}$$

$$+ \frac{f_2\,(2)}{2\,!} f^{k-2}(1)\ \{2\,k - 1 + k\,(k-1) \log\, f\,(1)\Big\},\ \ldots\Big],\ \ldots \qquad (15{\cdot}6)$$

by (8·1) (10·3)

§ 16. *The integral calculus.*

As in § 15, we may introduce the process of integration in the following manner:

$$\gamma\,[f(x, n)] + [C\,(n)] = \int \gamma'\,[f(x, n)]\,.\,d\,[f(x, n)]$$

$$= \int \gamma'\,[f(x, n)]\,.\,\left[\frac{d\,f\,(x, n)}{dx}\right] dx = \int [G\,(x, n)]\,dx, \text{ (say)},$$

$$= \left[\int G\,(x, n)\,dx\right], \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (16.1)$$

[C (n)] being an arbitrary constant A. F., or we may use the inverse process to that employed in (15.1). For the purpose of identifying the two results, use (15.5).

The theory of definite integration with arithmetical functions as limits is probably of interest, and has eventually a direct application to geometrical problems of great complexity.

(*To be continued.*)

SINGULARITIES OF PEDAL CURVES.

A. NARASINGA RAO, M.A., *Presidency College. Madras.*

The pedal of a curve with respect to a point (or origin) is usually defined as the locus of the foot of the perpendicular from the point, on a variable tangent to the curve. The inverse of the pedal with respect to a circle having its centre at the origin is the reciprocal polar of the curve with respect to the circle. The pedal is thus the inverse of the reciprocal polar. Rrciprocation changes point singularities into line singularities and *vice versa*. Since a point inverts into a point and a line into a circle, point singularities, in general, remain the same by inversion, while line singularities are altered.

§ 2. The effect of inversion on singularities is best studied by replacing the circle by a conic, called the base, and the origin by any point, not necessarily the centre of the conic. Points collinear with the origin and conjugate with respect to the base may be said to be inverse. This process is merely a generalised circular inversion known as *quadric inversion* and establishes a correspondence of point to point and straight line to conic.

Let O be the origin and OI, OJ, tangents to the base. The inverse of any point is determinate except when the point is O, I or J in which case the inverse is any point on the line I J, O I or O J. These points are therefore called the special points of the transformation, and the lines, the special lines. The degree of the inverse of a curve of class n is $2n$, except when the curve passes through a special point, in which case the inverse degenerates. For every point of intersection of the curve with a special line the corresponding special point is repeated in the inverse, that is to say, the inverse has nodes of the nth order at the special points.* It will be seen that contact with a special line gives rise to a cusp at the corresponding special point, and a node on the special line becomes a tacnode. If the tangent to the curve at a point of inter-

* The multiple points at O, I, J are the only singularities due to the special nature of the transformation. For a curve of degree $2n$ contains $\frac{2n(2n+3)}{2}$ constants and can therefore satisfy as many conditions. But a node of order n at a given point is equivalent to $1+2+3+\ldots n = \frac{n(n+1)}{2}$ conditions. With three such nodes. the number of disposable constants is $\frac{2n(2n+3)}{2} - \frac{3n(n+1)}{2}$ $= \frac{n(n+3)}{2}$ which is just the number of conditions that a curve of degree n can satisfy.

section Z, with the special line I J passes through O, the inverse has a point of inflexion at O with O Z for the inflexional tangent. An account of the singularities of inverse curves is given in detail in Miss Scott's *Modern Analytical Geometry* to which the reader is referred.

§ 3. By reciprocation, we obtain at once the following results :—

(1) The pedal transformation* is a case of correspondence of point to conic and of straight line to point.

(2) The pedal of a curve of class m is of order $2m$ with nodes of order m at the special points.

(3) To a multiple tangent corresponds in the pedal a multiple point of the same order.

(4) For every passage of a curve through a special point, the pedal developes a cusp at that point.

(5) Nodes on a special line transform into multiple tangents of the same order passing through the corresponding special point.

§ 4. Other results may be obtained in the same way, Thus†, where the curve has inflexional contact with the line I J, the inverse curve does not differ in appearance from an ordinary branch, though there is really a triple point at O. Hence when the curve has a cusp at O, the pedal has two vanishing loops at O, looking like an ordinary point, perhaps with a certain sharpness of bend as shown in figure 1. Again‡, the inverse of a cusp with the line I J as tangent, has at O a triple point with two evanescent loops but looking like an ordinary point of inflextion. Hence the pedal of a curve having an inflexion at O has really a triple point at O, though this appears just like an ordinary point of inflexion. Figure 2 exhibits the loop just before disappearance and also the final form on the pedal.

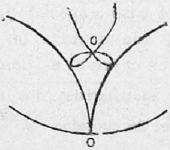


FIG. 1.

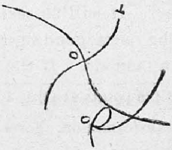


FIG. 2.

* The most general quadratic relation connecting points and line co-ordinates in two planes and birational in both sets of variables may be shown to be a generalised pedal transformation on the lines given in § 230, pages 217—19, Scott's *Analytical Geometry*.

† § 240, Scott's *Analytical Geometry*.

‡ Do. do.

§ 5. *Plucker's constants for the Pedal.*

Let us denote by o, i, j, the number of times a curve passes through O, I, J respectively and by o', i', j' their reciprocal singularities, *viz.* the number of times that it touches IJ, OI, OJ. Hence with the usual notation, (the capitals referring to the pedal) we have

$$N = 2m - o' - i' - j'$$

$$\Delta = T + \sum_{o', i', j'} \frac{(m - i' - j')(m - i' - j' - 1)}{2}$$

$$K = i + o + i' + j',$$

and the others (M, I & T) could be found in terms of these.

§ 6. A table is appended giving a list of simpler cases occurring in the usual Pedal transformation. The pedals of an ellipse, hyperbola and parabola are given in figures 3, 4 and 5.

No.	Singularity on The Curve C.	Singularity on The Pedal C′	REMARKS.
1	One pt. P at infinity ...	The asymptote to C at P is normal to the Pedal at the corresponding pt. P′	
2	Node ...	Circle through O touching the Pedal at two distinct points.	
3	Cusp at P. ...	Circle of curvature at the correspon-ding pt. P′ passes thro′ O.	
4	Multiple point of order r...	Circle through O touching the Pedal at r points	
5	Parallel tangents to O ... (Multiple point at infinity)	Multiple tangent thro′ O	The circle corresponding to (4) degenerates into a straight line.

No.	The Curve C.	The Pedal C.	REMARKS.
6	Double tangent ...	Node.	
7	Stationary tangent ...	Cusp.	
8	Class m. ...	Order 2 m.	
		Multiple points of order m at O and the circular points eqnivalent to $\frac{3}{2}m(m-1)$ nodes.	
	No. of Double tangents T.	nodes T.	
	No. of stationary tangents i. ...	cusps i.	
	Deficiency. $\frac{1}{2}(m-1)(m-2) - T - i$. ...	$D = \frac{1}{2}(2m-1)(2m-2) - 3\frac{m}{2}(m-1) - T - i = \frac{1}{2}(m-1)(m-2), - T - i$.	The deficiency of the Pedal is the same as that of C.
9	C passing through the circular points ...	Cusps at the circular points.	The pedal of a circle has cusps at the circular pts.
10	O on C.	Cusp at O.	
11	O on an asymptote of C.	O is a point of inflexion (Flecnode) on C', the asymptote being the normal at O.	Prove by forming the eqn. of the pedal of a conic w, r, t a point on an asymptote, *e.g.*, the pedal of a central conic with respect to its centre is $(x^2+y^2)^2 = a^2x^2 + b^2y^2$ has a biflecnode at the centre. (Page 171. Basset : *Cubic and Quartic Curves.*)

No.	The Curve C.	The Pedal C.	REMARKS.
12	O a focus.	The pedal curve degenerates into the circular lines thro' O and a residual curve of degree two less.	The pedal of a conic *w*, *r*, *t* a focus is the auxiliary circle and the line pair OI, OJ, which together constitute a bicircular quartic with a node at O.

§ 7. *Pedals of Conics.*

It follows from the preceding that the pedal of a central conic, is a rational bi-circular quartic, with a node at O. If O be the centre, this is a

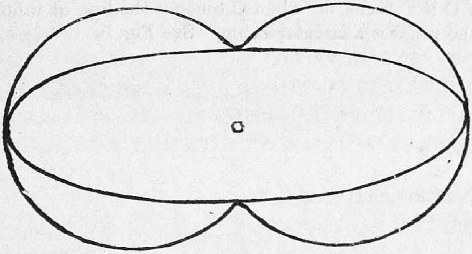


FIG. 3.

biflecnode, real for the hyperbola and imaginary for the ellipse. The pedal touches the conic at the feet of the 4 normals from O and has in

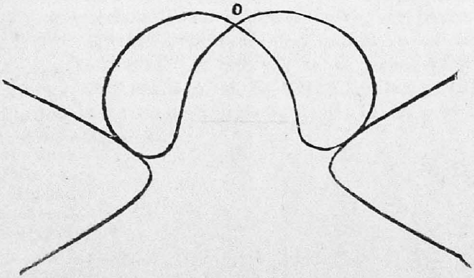


FIG. 4.

consequence no other point in common with it. When the conic is a circle, the nodes at I and J are replaced by cusps, and there is also a cusp at O if O be on the conic (*cf.* cardioid). The pedal degenerates

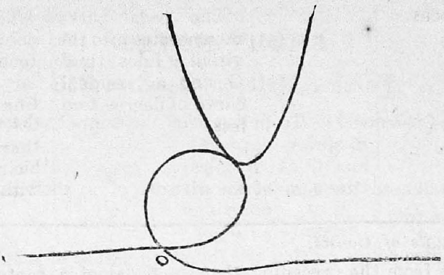


FIG. 5.

either when O is a focus or when C touches the line at infinity. In the latter case the pedal is a circular cubic. See Fig. 5.

SHORT NOTES.

Further Note on Question 653.

Q. 653. (Selected) :—To find a cube the sum of whose divisors is a perfect square.

Let σ (n) denote the sum of the divisors of n, including 1 and n we have to solve σ $(x^3) = y^2$.

I have succeeded in getting the following values of x and y.

	x	y
(1)	7.	20.
(2)	2.3.5.13.41.47.	$2^7.3^2.5^2.7.13.17.29.$
(3)	2.3.5.13.17.31.41.191.	$2^{12}.3^3.5^2.7.13.17.29^2.37.$
(4)	$2^3.3.5.11.19.31.43.743.$	$2^{12}.3^2.5^3.11.13.31.37.61.181.$
(5)	$2^5.5.23.47.211.241.577.$	$2^9.3^2.5^2.11.13^2.17^2.53.113.197.257.$
(6)	$2^5.5.17.23.31.191.211.241.577.$	$2^{14}.3^3.5^2.11.13^2.17^2.29.37.53.113.197.$ 257.
(7)	3.13.17.23.31.83.191.239.	$2^{16}.3^3.5^3.7.13^3.17.29.37.53.$
(8)	13.17.23.31.41.43.47.83.307.	$2^{15}.3^3.5^5.7^2.11.13^2.17.29^2.37.53.$
(9)	3.13.23.47.83.239.	$2^{11}.3^2.5^3.7.13^3.17.53.$
(10)	3.17.31.41.43.239.307.	$2^{17}.3^2.5^4.7.11.13^3.29^2.37.$ [257.
(11)	3.31.43.47.211.241.659.787.	$2^{14}.3.5^3.11^2.13.17.37.53.113.197.241.$

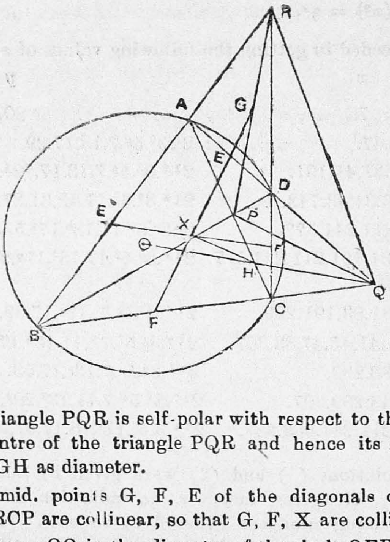
Of these, solutions ( ) and (2) were given by Lucas (*Theorie de Nombres*, p. 380). He gives, however, no indication as to how he got them. The remaining nine are believed to be new. [For a method of obtaining solution (1) see J. I. M. S., Vol. XI, (Dec. 1919), page 228]. If in (7), (9), (10), (11) we multiply x by $2 . 3^2 . 11$ and y by $2 . 3 . 11 . 61$, we get four other solutions.

N. B. MITRA.

Note on Question 1122.

Q. 1122. (V. Ramaswami Aiyar):—If the joins of four concyclic points A, B, C, D taken in opposite pairs intersect in P, Q, R, prove that the nine-point circle of PQR passes through the centroid of the points A, B, C, D.

Let P, Q, R be the points of intersection of (AC, BD), (AD, BC), (AB, CD) respectively; O the centre of the circle ABCD; E, F, G, H, X the mid. points of AD. BC, PR, OQ, EF respectively. Then, obviously, X is the centroid of the quadrilateral ABCD.



Since the triangle PQR is self-polar with respect to the circle ABCD, O is the orthocentre of the triangle PQR and hence its nine-point circle is the circle on GH as diameter.

Now, the mid. points G, F, E of the diagonals of the complete quadrilateral BROP are collinear, so that G, F, X are collinear.

Again, because OQ is the diameter of the circle QEF and X the mid. point of the chord EF, HX is perpendicular to XF, *i.e.*, $H\hat{X}G$ is right.

Hence X lies on the circle with HG for diameter, *i.e.* on the nine-point circle of the triangle PQR.

Cor. 1. In the cyclic quadrilateral ABCD, the perpendiculars from the mid. point of each side on the opposite side are concurrent, the point of concurrence being on the circum-circle of the triangle PQR.

Join OX and produce it to O′ such that O′X = XO.

Then O′ is a point on the circum-circle of the triangle PQR.

Again, as EF and OO′ bisect each other, O′E is parallel to OF and therefore perpendicular to BC, and O′F is parallel to OE and therefore perpendicular to AD.

Thus the perpendiculars from the mid-points of AD, BC on the opposite sides BC, AD meet at O′ on the circum-circle of triangle PQR.

Similarly by joining the mid-points E′, F′ of AB, CD to O′ it can be shown that these joins are perpendicular to CD, AB respectively.

Hence the proposition.

2. Incidentally, we may note that the point of concurrence O′ of the four perpendiculars is the common ortho-centre of the two triangles QEF and RE′F′.

3. When the diagonals AC, BD cut at right-angles, it is obvious that the four perpendiculars pass through the point P which is the intersection of the diagonals. Thus in this case O′ coincides with P.

A. A. KRISHNASWAMI AIYANGAR.

Astronomical Notes.

1. *Nova Cygni III* :—One of the most important astronomical events of the year 1920 was the appearance of a new star in the Constellation Cygnus. The star was first discovered by Mr. Denning at Bristol on the night of 20th August. The magnitude at that time was 3·5, and since then, the brightness increased rapidly (like all other Novæ) ; the maximum (magnitude 2·2) was reached on August 24. The decrease in brightness was rather more rapid than that of some of the Novæ which appeared recently. Dr. Lockyer has estimated the decline to be on the average five times more rapid than that of Nova Aquilæ (1918) or Nova Persei (1901).

The position of the Nova is

R. A. 19^h 56^m and Decl. 53° 24′ North.

Forty-one new stars have been discovered to have made their appearance till now, since Tycho Brahe's star of 1572, excluding the faint novæ which have appeared in the midst of Nebulæ and Clusters ; the number recorded during the last thirty years alone is about thirty-one per annum on the average.

2. *Pons-Winnecke's Comet* :—This is one of the short period comets which are due to perihelion during 1921. The last observed return was on Sep. 1, 1915 and, its period being a little over 5½ years the next return will happen about the end of June. As the comet will be very near the earth at that time, the return is likely to be interesting and we may probably be treated to a brilliant meteoric shower.

3. *Saturn* :—This planet has been under continuous observation recently, the interest attached to the occasion being due to the fact that the earth passed through the plane of his rings in Nov. 1920. This happens once every fifteen years and at the time of passage, the minor axis of the ring ellipse continues diminishing slowly and the rings themselves remain invisible for some short time. The actual passage occurred on Nov. 7, 1920, and the phenomenon was well observed by several astronomers. Prof. Barnard estimates that the rings are probably not more than fifty miles thick. For an interesting discussion of the condition of the earth passing through the plane of Saturn's rings, reference may be made to Prof. Turner's paper in the *Monthly Notices of the R.A.S.* 1908 April. The problem is also briefly dealt with in Ball's *Spherical Astronomy*, Ex. 23 page 428.

NIZAMIAH OBSERVATORY,
HYDERABAD.

T. P. BHASKARA SASTRI.

SOLUTIONS.

Question 1059.

(K. J. SANJANA, M.A.) :—Prove that the sums of the products of the quantities $(2^2-1)^{-1}$, $(4^2-1)^{-1}$, $(6^2-1)^{-1}$ ...... ad inf., taken three at a time and four at a time, are respectively $\frac{10-\pi^2}{32}$ and $\frac{1680-180\pi^2+\pi^4}{6144}$; and show how to find the sums of the products taken five, six .... at a time.

Solutions by Hemraj, N. G. Leather and G. S. Mahajani.

$$\frac{\sin \pi y/2}{\sin \pi z/2} = \frac{y}{z} \prod_{1}^{\infty} \frac{(2r)^2-y^2}{(2r)^2-z^2} = \frac{y}{z} \Pi \left((1 + \frac{x}{(2r^2)-z^2} \right) \text{ where}$$

$y^2 = z^2 - x$. Put $z = 1$ and we get

$$(1-x)^{-\frac{1}{2}} \sin\frac{\pi}{2} (1-x)^{\frac{1}{2}} = \Pi \left(1 + \frac{x}{(2r)^2-1}\right) = 1 + \sum_{1}^{8} A_r \; x^r \text{ ...... (1)}$$

where A_r denotes the sum of the products of $(2^2-1)^{-1}$, $(4^2-1)^{-1}$ ... taken r at a time.

$$\text{Now } (1-x)^{-\frac{1}{2}} = 1 + \tfrac{1}{2} x + \frac{1\cdot3}{2\cdot4} x^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6} x^3 + \text{..........}$$

$$\sin \frac{\pi}{2} (1-x)^{\frac{1}{2}} = \sin \left[\frac{\pi}{2} \left(1 - \frac{1}{2} x - \frac{1}{2^2} \frac{x^2}{2!} - \frac{3}{2^3} \frac{x^3}{3!} - \frac{15}{2^4} \frac{x^4}{4!} \cdots \right) \right]$$

$$= \sin \left(a + b x + c \frac{x^2}{2!} + d \frac{x^3}{3!} + e \frac{x^4}{4!} + \text{........} \right)$$

$$= A + B x + C \frac{x^2}{2!} + D \frac{x^3}{3!} + E \frac{x^4}{4!} + \text{........}$$

By Arbogast's Method of Derivations (Art 88, *Diff.* : *Cal.* : Williamson).

$A = \sin a$, $B = \delta. \sin a = b \cos a$, $C = \delta. b \cos a = c \cos a - b^2 \sin a$ etc.

But $a = \frac{\pi}{2}$, $\therefore A = 1$, $B = 0$, $C = -\frac{\pi^2}{16}$, $D = -\frac{3\pi^2}{32}$,

$$E = -\frac{15\pi^2}{64} + \frac{\pi^4}{256} \text{ etc.}$$

$$\text{Hence } (1-x)^{-\frac{1}{2}} \sin \left[\frac{\pi}{2} (1-x)^{\frac{1}{2}}\right]$$

$$= \left\{1 + \tfrac{1}{2} x + \frac{1\cdot3}{2\cdot4} x^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6} x^3 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8} x^4 + \text{......} \right\}$$

$$\left\{1 - \frac{\pi^2}{32} x^2 - \frac{\pi^2}{64} x^3 - \left(\frac{15\pi^2}{64} - \frac{\pi^4}{256}\right) \frac{x^4}{4!} \text{........} \right\}$$

$$= 1 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots\ldots + A_r x^r + \ldots\ldots \text{ from (1)}$$

Comparing co-efficients of x^r on both sides, we get A_r.

In particular $A_1 = \frac{1}{2}$; $A_2 = \dfrac{12 - \pi^2}{32}$; $A_3 = \dfrac{10 - \pi^2}{32}$;

$$A_4 = \frac{1680 - 180\pi^2 + \pi^4}{6144}.$$

Question 1080.

(K. J. Sanjana, M.A.) :—Show how to find the sum of the products r at a time of the quantities $(3^2-1)^{-1}$, $(5^2-1)^{-1}$, $(7^2-1)^{-1}$, ... ad inf.; and find the values for $r = 3$ and $r = 4$.

Solutions by Hemraj, N. G. Leather and G. S. Mahajani.

$$\frac{\cos \pi y/2}{\cos \pi z/2} = \prod_1^\infty \left[1 + \frac{x}{(2n-1)^2 - z^2}\right], \text{ where } y^2 = z^2 - x.$$

Put $z = 1$ and we get

$$\frac{4}{\pi x} \cos \frac{\pi}{2} (1 - x)^{\frac{1}{2}} = \left(1 + \frac{x}{3^2-1}\right) \left(1 + \frac{x}{5^2-1}\right)$$

$$\left(1 + \frac{x}{7^2-1}\right) \ldots\ldots\ldots$$

$$= 1 + \sum_1^\infty B_r x^r \text{ where } B_r \text{ stands for the sum of}$$

the products of $(3^2 - 1)^{-1}$, $(5^2 - 1)^{-1}$ etc., taken r at a time.

$$\cos \frac{\pi}{2} (1-x)^{\frac{1}{2}} = \cos \frac{\pi}{2}\left[\left(1 - \frac{1}{2} x - \frac{1}{2^2}\frac{x^2}{2!} - \frac{3}{2^3}\frac{x^3}{3!} \ldots\ldots\ldots\right)\right]$$

$$= \cos \left(a + b x + c\frac{x^2}{2!} + d\frac{x^3}{3!} + e\frac{x^4}{4!} + \ldots,\right.$$

$$= A + Bx + C\frac{x^2}{2!} + D\frac{x^3}{3!} + E\frac{x^4}{4!} + F\frac{x^5}{5!} + \ldots$$

By Art. 88, *Diff. Cal.*: Williamson, we have

$$A = 0;\ B = -b = \frac{\pi}{4};\ C = -c = \frac{\pi}{8};$$

$$D = -d + b^3 = \frac{3\pi}{16} - \frac{\pi^3}{64};\ E = -e + 6b^2c = \frac{15\pi}{32} - \frac{3\pi^3}{64}$$

$$F = -f + 5(2b^2d + 3bc^2) - b^5 = \frac{105\pi}{64} - \frac{135}{256}\pi^3 + \frac{\pi^5}{1024}, \text{ etc.}$$

$$\therefore \frac{4}{\pi x} \cos \frac{\pi}{2} (1-x)^{\frac{1}{2}} = 1 + \tfrac{1}{4} x + \frac{12 - \pi^2}{96} x^2 + \frac{10 - \pi^2}{128} x^3 + \frac{1680 - 540\pi^2 + \pi^4}{30720} x^4 + \ldots$$

$$= 1 + B_1 x + B_2 x^2 + \ldots + B_r x^2 + \ldots$$

Equating co-efficients of x^r on both sides, we get B_r.

In particular $B_1 = \frac{1}{4}$; $B_2 = \dfrac{12 - \pi^2}{96}$; $B_3 = \dfrac{10 - \pi^2}{128}$;

$$B_4 = \frac{1680 - 540\pi^2 + \pi^4}{30720}.$$

Question 1062.

(C. KRISHNAMACHARI) :—Prove that

$$1 + \left({}_{n-1}C_1\right) \frac{1}{3} + \left({}_{n-1}C_2\right) \frac{1}{5} + \ldots \frac{1}{2n-1}$$

$$= 2^{n-1} \left\{ \frac{1}{2n-1} + \frac{(n-1)}{(2n-1)(2n-3)} + \ldots + \frac{(n-1)!}{(2n-1)(2n-3)\ldots 1} \right\}.$$

Solutions by N. G. Leather, Vasavada and Kewalramani.

$$1 + {}_{n-1}C_1 \frac{1}{3} + {}_{n-1}C_2 \frac{1}{5} + \ldots + \frac{1}{2n-1} = \int_0^1 (1 + x^2)^{n-1} dx$$

$$= \int_0^{\pi/4} \sec^{2n} \theta \, d\theta$$

$$= \Big[\sec^{2n-2} \theta \tan \theta \Big]_0^{\pi/4} - (2n-2) \int_0^{\pi/4} \sec^{2n-2} \theta \tan^2 \theta \, d\theta$$

$$= 2^{n-1} - (2n-2) \int_0^{\pi/4} (\sec^{2n} \theta - \sec^{2n-2} \theta) \, d\theta$$

$$= \frac{1}{2n-1} \left\{ 2^{n-1} + 2(n-1) \int_0^{\pi/4} \sec^{2n-2} \theta \, d\theta \right\}$$

$$= \left[\frac{2^{n-1}}{2n-1} + \frac{2^{n-1}}{2n-1} \frac{(n-1)}{.\,2n-3} + \frac{2^{n-1}}{(2n-1)} \frac{(n-1)(n-2)}{(2n-3)(2n-5)} + \ldots + \frac{2^{n-1} (n-1)!}{(2n-1)(2n-3)\ldots 1} \int_0^{\pi/4} \sec^2 \theta \, d\theta \right]$$

$$= 2^{n-1} \left[\frac{1}{2n-1} + \frac{n-1}{2n-1 \,.\, 2n-3} + \frac{(n-1)(n-2)}{2n-1 \,.\, 2n-3 \,.\, 2n-5} + \ldots \frac{(n-1)!}{(2n-1)(2n-3)\ldots 3 \,.\, 1} \right].$$

Question 1088.

(C. Krishnamachari) :—Show that

(a) $\frac{1}{2^{n+1}}\left\{\frac{1}{n}+\frac{(2n-1)}{n(n-1)}+\dots\frac{(2n-1)(2n-3)\dots 3}{n(n-1)\dots 1}\cdot\frac{\pi}{4}\right\}$

$$= 1-(n+1)\frac{1}{3}+\frac{(n+1)(n+2)}{1\,.\,2}\cdot\frac{1}{5}-\dots$$

(b $1-\binom{n-\frac{1}{2}}{1}\frac{1}{3}+\binom{n-\frac{1}{2}}{2}\frac{1}{5}\dots$

$$= 2^{n+\frac{1}{2}}\left[\frac{1}{2n+1}-\frac{1}{2}\binom{n-\frac{1}{2}}{1}\cdot\frac{1}{2n+3}+\dots\right].$$

Solutions by N. G. Leather, Vasavada, and Kewalramani.

(a) R. H. S. $=\int_0^1(1+x^2)^{-(n+1)}\,dx=\int_0^{\pi/4}\cos^{2n}\theta\,d\theta$

$$=\Big[\cos^{2n-1}\theta\sin\theta\Big]_0^{\pi/4}+(2n-1)\int_0^{\pi/4}\cos^{2n-2}\theta\sin^2\theta\,d\theta$$

$$=\frac{1}{2^n}+(2n-1)\int_0^{\pi/4}(\cos^{2n-2}\theta-\cos^{2n}\theta)\,d\theta$$

$$=\frac{1}{2n}\left[\frac{1}{2^n}+(2n-1)\int_0^{\pi/4}\cos^{2n-2}\theta\,d\theta\right]$$

$$=\frac{1}{2^{n+1}}\left[\frac{1}{n}+\frac{2n-1}{n\,.\,n-1}+\frac{2n-1\,.\,2n-3}{n\,.\,n-1\,.\,n-2}\dots\right.$$
$$\left.+\frac{2n-1\,.\,2n-3\,.\,2n-5\dots3\,.\,1}{n\,!}\cdot\int_0^{\pi/4}d\theta\right]$$

$$=\frac{1}{2^{n+1}}\left[\frac{1}{n}+\frac{2n-1}{n\,.\,n-1}+\frac{2n-1\,.\,2n-3}{n\,.\,n-1\,.\,n-2}+\dots\right.$$
$$\left.+\frac{2n-1\,.\,2n-3\,.\,2n-5\dots3\,.\,1}{n\,!}\cdot\frac{\pi}{4}\right].$$

(b) L. H. S. $=\int_0^1(1-x^2)^{n-\frac{1}{2}}\,dx=\frac{1}{2}\int_0^1(1-y)^{n-\frac{1}{2}}y^{-\frac{1}{2}}\,dy\ (y=x^2)$

$$=\frac{1}{2}\,\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(n+1)}.$$

The series on the right-hand side.

$$=2^{2n+1}\left[\frac{1}{2n+1}\left(\frac{1}{\sqrt{2}}\right)^{2n+1}-{}_{n-\frac{1}{2}}C_1\left(\frac{1}{\sqrt{2}}\right)^{2n+3}\frac{1}{2n+3}+\dots\right]$$

$$=2^{2n+1}\int_0^{\frac{1}{\sqrt{2}}}\left[x^{2n}-{}_{n-\frac{1}{2}}C_1\,x^{2n+2}+\dots\right]dx$$

$$= 2^{2n+1}\int_0^{\frac{1}{\sqrt{2}}} x^{2n}\left(1-x^2\right)^{n-\frac{1}{2}} dx$$

$$= 2^{2n+1}\int_0^{\pi/4} \sin^{2n}\theta\cos^{2n}\theta\, d\theta = 2^{2n}\int_0^{\pi/2}\sin^{2n}\theta\cos^{2n}\theta\, d\theta$$

$$= 2^{2n-1}\frac{\{\Gamma(n+\frac{1}{2})\}^2}{\Gamma(2n+1)} = \frac{2^{2n-1}\Gamma(n+\frac{1}{2})[(n-\frac{1}{2})(n-\frac{3}{2})\dots\frac{1}{2}]\Gamma(\frac{1}{2})}{2n!}$$

$$= \frac{1}{2}\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{n(n-1)(n-2)\dots 1} = \frac{1}{2}\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n+1)} = \text{L. H. S.}$$

Question 1065.

(S. Krishnaswami Aiyangar):—If ρ be the radius of curvature of the curve $r^m = a^m \sin m\theta$ at the point whose distance measured along the curve from a fixed point is s, prove that

$$(m-1)(m+1)^2\rho\frac{d^2\rho}{ds^2} - m(m+1)^2\left(\frac{d\rho}{ds}\right)^2 - m(m-1)^2 = 0.$$

Solution by P. R. Venkatakrishna Iyer and Kewalramani.

Using the notion given in Edward's *Diff. Calculus*, for the curve $r^m = a^m \sin m\theta$,

$$\phi = m\theta \text{ and } \psi = (m+1)\theta \text{. and}$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2\operatorname{cosec}^2 m\theta = a^2(\sin m\theta)^{\frac{2}{m}-2}$$

$$= a^2\left(\sin\frac{m}{m+1}\psi\right)^{\frac{2}{m}-2}.$$

$$\therefore \quad \frac{ds}{d\theta} = a\left(\sin\frac{m}{m+1}\psi\right)^{\frac{1}{m}-1}.$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\Big/\frac{d\psi}{d\theta} = \frac{1}{m+1}\cdot\frac{ds}{d\theta}$$

$$= \frac{a}{m+1}\left(\sin\frac{m}{m+1}\psi\right)^{\frac{1}{m}-1}.$$

Taking the logarithmic differential of ρ with respect to s, we have

$$\frac{d\rho}{ds} = -\frac{m-1}{m+1}\cot\frac{m}{m+1}\psi. \quad \dots \quad \dots \quad (1)$$

Differentiating (1) we get

$$\rho\cdot\frac{d^2\rho}{ds^2} = \frac{(m-1)m}{(m+1)^2}\operatorname{cosec}^2\frac{m}{m+1}\psi. \quad \dots \quad (2)$$

Eliminating ψ between (1) and (2), we get the result

$$(m-1)(m+1)^2\rho\frac{d^2\rho}{ds^2}-m(m+1)^2\left(\frac{d\rho}{ds}\right)^2-m(m-1)^2=0.$$

Now to solve the differential equation, we can write it in the form

$$a\rho\frac{d^2\rho}{ds^2}-b\left(\frac{d\rho}{ds}\right)^2-c=0.$$

$$\frac{d\rho}{ds}=\left(\frac{ds}{d\rho}\right)^{-1}\text{ and }\frac{d^2\rho}{ds^2}=\left(\frac{ds}{d\rho}\right)^{-1}\cdot\frac{d}{d\rho}\left(\frac{ds}{d\rho}\right)^{-1}=-\frac{d^2s}{d\rho^2}\Big/\left(\frac{ds}{d\rho}\right)^3.$$

Substituting these in the differential equation,

we get $$a\rho\cdot\frac{d^2s}{d\rho^2}+b\frac{ds}{d\rho}+c\left(\frac{ds}{d\rho}\right)^3=0.$$

In this, put $p=\dfrac{ds}{d\rho}$, then $\dfrac{d^2s}{d\rho^2}=\dfrac{dp}{d\rho}$.

Thus we have $a\rho\cdot\dfrac{dp}{d\rho}+bp+cp^3=0,$

i.e., $$\frac{a}{b}\left\{\frac{1}{p}-\frac{cp}{cp^2+b}\right\}dp+\frac{d\rho}{\rho}=0.$$

Integrating, we get

$$\frac{a}{b}\log\frac{p}{\sqrt{cp^2+b}}+\log\rho=\text{a constant}=k$$

$$\therefore\quad\frac{p^2}{cp^2+b}=k^2\rho^{-\frac{2b}{a}}.$$

$$\therefore\quad p^2=\frac{bk^2\rho^{-\frac{2b}{a}}}{1-ck^2\rho^{-\frac{2b}{a}}},$$

i.e. $$\frac{ds}{d\rho}=\frac{b^{\frac{1}{2}}k\rho^{-\frac{b}{a}}}{\sqrt{1-ck^2\rho^{-\frac{2b}{a}}}}.$$

Integrating $$s=k'+\int\frac{b^{\frac{1}{2}}k\rho^{-\frac{b}{a}}}{\sqrt{1-ck^2\rho^{-\frac{2b}{a}}}}\,d\rho.$$

Question 1067.

(V. TIRUVENKATACHARI): Show that

$$\int_0^{\pi/2} \frac{\log \sin x . \log \cos x}{\sqrt{\sin x \cos x}}\, dx = \tfrac{1}{8}\, \frac{\{\Gamma(\tfrac{1}{4})\}^2}{\sqrt{\pi}} \left\{ (\log 2)^2 + \pi \log 2 - \frac{\pi^2}{4} \right\}.$$

Solution by Hemraj and Kewalramani.

$$\int_0^{\pi/2} \sin^{2\alpha-1} x \cos^{2\beta-1} x\, dx = \tfrac{1}{2}\, \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Differentiate 1st with respect to α and then with respect to β

$$4 \int_0^{\pi/2} \sin^{2\alpha-1} x \cos^{2\beta-1} x \log \sin x . \log \cos x\, dx$$

$$= \tfrac{1}{2}\, \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \{\psi(\alpha) - \psi(\alpha+\beta)\} \left\{ \psi(\beta) - \psi(\alpha+\beta) - \frac{\psi'(\alpha+\beta)}{\psi(\alpha) - \psi(\alpha+\beta)} \right.$$

where $\psi(x) = \dfrac{d}{dx}\{\log \Gamma(x)\}$.

Put $\alpha = \beta = \frac{1}{4}$ and we get

$$4 \int_0^{\pi/2} \frac{\log \sin x \log \cos x}{\sqrt{(\sin x \cos x)}}\, dx = \frac{\{\Gamma(\tfrac{1}{4})\}^2}{2\sqrt{\pi}} \left\{ [\psi(\tfrac{1}{4}) - \psi(\tfrac{1}{2})]^2 - \psi'(\tfrac{1}{2}) \right\}$$

$$= \frac{\{\Gamma(\tfrac{1}{4})\}^2}{2\sqrt{\pi}} \left\{ (\log 2)^2 + \pi \log 2 - \frac{\pi}{4} \right\} \text{from}$$

Bromwich: *Theory of Infinite Series*, p. 475 Exs. 42 and 43.

Question 1069.

(A. C. L. WILKINSON):—Normals PA, PB, PC, PD are drawn to a hyperbola; PA meets the hyperbola again in A′, PB in B′, PC in C′, PD in D′. If A′B′C′D′ are concyclic, prove that the locus of P is an ellipse of which the equi-conjugate diameters coincide with the asymptotes of the hyperbola.

Solution by Hemraj.

The equation of the hyperbola referred to its asymptotes as axes is $xy = c^2$ and the normal at (t) is $y(1 - t^2 \cos \omega) - x(t^2 - \cos \omega) = \frac{c}{t}(1 - t^4)$ where ω is the angle between the asymptotes.

It passes through the point P(x, y) if

$$ct^4 - (y \cos \omega + x)\, t^3 + (y + x \cos \omega)\, t - c = 0.$$

$$\therefore \Sigma t_1 = \frac{y \cos\omega + x}{c}, \Sigma \frac{1}{t_1} = \frac{y + x\cos\omega}{c}, t_1 t_2 t_3 t_4 = -1 \quad \ldots (1)$$

$$\Sigma t_1 t_2 = 0 \quad \therefore \Sigma t_1^2 t_2^2 = 2 \Sigma t_1 \Sigma \frac{1}{t_1} - 2 \,;\ (\Sigma t_1)^2 = \Sigma t_1^2 \,;$$

$$\left(\Sigma \frac{1}{t_1}\right)^2 = \Sigma \frac{1}{t_1^2} \quad \ldots (2)$$

Hence from (1)

$$x^2+y^2 = \frac{c^2}{\sin^4\omega}\left\{(1+\cos^2\omega)\left[\Sigma t_1^2 + \Sigma \frac{1}{t_1^2}\right] - 4\cos\omega\, \Sigma t_1 \Sigma \frac{1}{t_1}\right\} \ldots (3)$$

where A, B, C, D are t_1, t_2, t_3, t_4.

The normal meets the hyperbola again in a point (T) where

$$T = \frac{-t(t^2 - \cos\omega)}{1 - t^2\cos\omega} \text{ and (T) lies on a circle}$$

$$x^2 + y^2 + 2xy\cos\omega + 2gx + 2fy + k = 0 \text{ if}$$

$$c^2 T^4 + 2fc\, T^3 + (2c^2\cos\omega + k)\, T^2 + 2gc\, T + c^2 = 0.$$

Hence if A′, B′, C′, D′ be T_1, T_2, T_3, T_4, then $T_1 T_2 T_3 T_4 = 1$

$$\therefore t_1(t_1^2 - \cos\omega) \,.\, t_2(t_2^2 - \cos\omega) \,.\, t_3(t_3^2 - \cos\omega) \,.\, t_4(t_4^2 - \cos\omega)$$

$$= (1 - t_1^2\cos\omega)(1 - t_2^2\cos\omega)(1 - t_3^2\cos\omega)(1 - t_4^2\cos\omega),$$

i.e., $$\cos\omega(1+\cos^2\omega)\left[\Sigma t_1^2 + \Sigma\frac{1}{t_1^2}\right] - 4\cos^2\omega\left[\Sigma t_1 \Sigma \frac{1}{t_1} - 1\right]$$

$$= 2(1 + \cos^4\omega), \text{ from (1) and (2).}$$

Substituting in (3) we get

$$x^2 + y^2 = \frac{2c^2}{\sec\omega}$$

which is an ellipse referred to its equi-conjugate diameters as axes.

Question 1071.

(T. P. Trivedi) :—If $S_r = \sum_{p=1}^{p=r+1} p^{r+1-p}\ {}_{n+1-p}C_{r+1-p}$

prove that $\sum_{r=0}^{n} S_r = 1^{n+1} + 2^n + 3^{n-1} + \ldots + (n+1)$.

Solution by Hemraj and K. B. Madhava.

Obviously

$$S_r = 1^r \,.\, {}_nC_r + 2^{r-1} \,.\, {}_{n-1}C_{r-1} + 3^{r-2} \,.\, {}_{n-2}C_{r-2} + \ldots$$

$$\therefore \sum_{r=0}^{n} S_r = \sum_{r=0}^{n} {}_nC_r + \sum_{r=0}^{n-1} 2^r \, {}_{n-1}C_r + \sum_{r=0}^{n-2} 3^r \cdot {}_{n-2}C_r + \dots$$

$$= (1+1)^n + (1+2)^{n-1} + (1+3)^{n-2} + \dots$$

$$= 2^n + 3^{n-1} + 4^{n-2} + \dots (n+1) + (n+2)^0.$$

$$= 1^{n+1} + 2^n + 3^{n-1} + \dots (n+1)$$

for the sake of symmetry.

Question 1072.

(T. P. TRIVEDI, M.A., LL.B.) :—Prove that

$$\int_0^{\pi/2} \sqrt{\tan x} \log \tan x \, dx = \frac{\pi^2}{2\sqrt{2}}.$$

$$\int_0^{\pi/2} \sqrt[5]{\tan x} \log \tan x \, dx = \frac{\pi^2}{4} \left(\frac{3}{\sqrt{5}} - 1 \right).$$

Solutions by Hemraj and M. K. Kewalramani.

We know that $\int_0^{\infty} \frac{y^{k-1}}{1+y} dy = \pi \operatorname{cosec} k\pi$, where $1 > k > 0$.

Put $y = \tan^2 x$ and differentiate with respect to k :

$$4 \int_0^{\pi/2} \tan^{2k-1} x \log \tan x \, dx = -\pi^2 \operatorname{cosec} k\pi \cdot \cot k\pi.$$

Putting $k = \frac{3}{4}$ and $\frac{3}{5}$ respectively, we get the results ; the first result correctly printed is therefore $\frac{\pi^2}{2\sqrt{2}}$.

Question 1080.

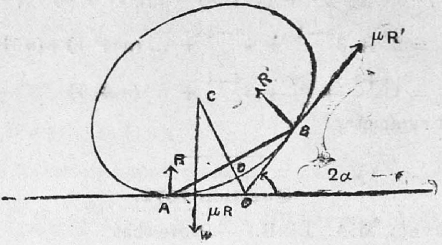
(K. J. SANJANA, M.A.) :—An elliptic cylinder of eccentricity e rests on a horizontal plane and against a plane inclined at an angle 2α to the horizon, touching them along lines parallel to and equidistant from their line of intersection. If the coefficient of friction (μ) be the same for each plane and the cylinder be just on the point of slipping, prove that μ is the positive root of the equation

$$\mu^2(\tan \alpha - e^2 \sin \alpha \cos \alpha) + \mu = e^2 \sin \alpha \cos \alpha.$$

Solution by Hemraj.

Take a section by a vertical plane.

Since the tangents OA and OB are equal, O lies on either axis. But



it is easy to show that in this case it lies on the minor axis. Take O (o, y_1), then the equation of AB is $yy_1 = b^2$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ being that of the ellipse.

$$\therefore \cot \alpha = \frac{AD}{OD} = \frac{a}{\sqrt{y_1^2 - b^2}}, \text{ i.e., } y_1 = a\sqrt{(\sec^2 \alpha - e^2)}.$$

$$\therefore OA = OD \operatorname{cosec} \alpha = \frac{a \tan \alpha \sec \alpha}{\sqrt{\sec^2 \alpha - e^2}}.$$ [2α may be acute or obtuse].

Now the system of forces acting on the body is in limiting equilibrium,

Resolving horizontally and vertically and taking moments about O, we get,

$$\mu R = R' (\sin 2\alpha - \mu \cos 2\alpha) \quad \ldots \quad \ldots \quad (1)$$

$$W = R + R' (\mu \sin 2\alpha + \cos 2\alpha) \quad \ldots \quad \ldots \quad (2)$$

$$W . OC . \sin \alpha = (R - R') . OA \quad \ldots \quad \ldots \quad (3)$$

Substituting for R and R' in (3) from (1) and (2) we get

$$\sin \alpha . \sqrt{\sec^2 \alpha - e^2} = \frac{\sec \alpha (\sin \alpha - \mu \cos \alpha) \tan \alpha}{\sin \alpha (1 + \mu^2) \sqrt{\sec^2 \alpha - e^2}},$$

i.e., $\cos \alpha \sin \alpha (\sec^2 \alpha - e^2)(1 + \mu^2) = (\tan \alpha - \mu)$,

i.e., $\mu^2 (\tan \alpha - e^2 \sin \alpha \cos \alpha) + \quad = e^2 \sin \alpha \cos \alpha$

which gives μ.

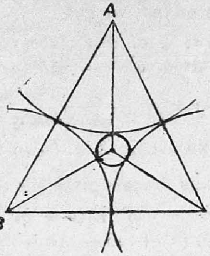
Question 1091.

(Enquirer) :—If a, b, c be the radii of three circles which touch one another externally, and r_1 and r_2 be the radii of the two circles that can be drawn to touch these three, prove that

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}$$

Solution and Remarks by K. J. Sanjana.

Let A, B, C be the centres of the respective circles, O the centre and



x the radius of the circle with which they have external contact. $\angle BAC = A$, we have

$$\cos A = \frac{a^2 + ab + ac - bc}{(a+b)(a+c)}, \quad \sin A = \frac{2\sqrt{abc(a+b+c)}}{(a+b)(a+c)}.$$

If $\angle OAB = \lambda$, $\angle OAC = \mu$, we have similarly

$$\cos \lambda = \frac{a^2 + ab + ax - bx}{(a+b)(a+x)}, \quad \cos \mu = \frac{a^2 + ac + ax - cx}{(a+c)(a+x)}.$$

But since $A = \lambda + \mu$, $\cos A = \cos\lambda \cos\mu - \sin\lambda \sin\mu$; transposing, squaring and simplifying, we get

$$\sin^2 A = \cos^2\lambda + \cos^2\mu - 2\cos A \cos\lambda \cos\mu.$$

Substituting and multiplying throughout by $(a+b)^2 (a+c)^2 (a+x)^2$, this gives

$$(a+c)^2 (a^2 + ab + ax - bx)^2 + (a+b)^2 (a^2 + ac + ax - cx)^2$$
$$- 2(a^2 + ab + ac - bc)(a^2 + ab + ax - bx)(a^2 + ac + ax - cx)$$
$$= 4abc(a+b+c)(a+x)^2.$$

This is a quadratic in x, which shows that the radius of the circle (with sign changed if necessary with which the given circles have internal contact is also obtained.

Simplifying, we get

$$x^2 \{ b^2c^2 + c^2a^2 + a^2b^2 - 2abc(a+b+c) \}$$
$$- 2abc(bc + ca + ab)x + a^2b^2c^2 = 0.$$

If r_1, r_2 are the two roots, $\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2}$

$$= \frac{2abc(bc + ca + ab)}{a^2b^2c^2} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

The proposer has taken the numerical value of the second root. The result obtained is that of Ex. 62, Chap. XII, Hobson's *Plane Trigonometry*, which has I think already been asked in our columns.

When the equation is solved, we get

$$x = \frac{abc\{bc + ca + ab \pm 2\sqrt{abc(a + b + c)}\}}{b^2c^2 + c^2a^2 + a^2b^2 - 2abc(a + b + c)};$$

on rationalizing, this becomes $\dfrac{abc}{bc + ca + ab \mp 2\sqrt{abc(a + b + c)}}$, which is the result of Ex. 29, Chap. XII. *loc. cit.*

For internal contact this process gives the equation

$$(a + c)^2 (a^2 + ab - ax + bx)^2 + (a + b)^2 (a^2 + ac - ax + cx)^2$$
$$- 2(a^2 + ab + ac - bc)(a^2 + ab - ax + bx)(a^2 + ac - ax + cx)$$
$$= 4abc(a + b + c)(x - a)^2,$$

which is the first equation with the sign of x changed.

The six distances of the four points A, B, C, O are a, b, c, $a \pm x$, $b \pm x$, $c \pm x$; and the formula connecting such distances may be employed. It is found to give lengthy expressions involving x^4 and x^3, the coefficients of which, however, vanish.

By the help of *Frobenius's Theorem*, Casey has deduced the following equation for the radius of a circle cutting three circles at angles ϕ_1, ϕ_2, ϕ_3 (Eqn. 276, *Analytical Geometry*):—

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma & \rho \\ \alpha & -1 & \cos\overline{12} & \cos\overline{13} & \cos\phi_1 \\ \beta & \cos\overline{21} & -1 & \cos\overline{23} & \cos\phi_2 \\ \gamma & \cos\overline{31} & \cos\overline{32} & -1 & \cos\phi_3 \\ \rho & \cos\phi_1 & \cos\phi_2 & \cos\phi_3 & -1 \end{vmatrix} = 0,$$

where I have put α, β, γ, ρ for the reciprocals of a, b, c, x as taken above. As $\cos\overline{12}$, &c. are each -1, and $\cos\phi_1$, &c. are each either -1 or $+1$, the determinant equation is found to reduce to

$$\rho^2 - 2\rho(\alpha + \beta + \gamma) + \alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta = 0.$$

Hence $\rho_1 + \rho_2$ or $\dfrac{1}{r_1} + \dfrac{1}{r_2} = 2\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$, and

$$\rho_1\rho_2 \text{ or } \frac{1}{r_1r_2} = \frac{b^2c^2 + c^2a^2 + a^2b^2 - 2abc(a + b + c)}{a^2b^2c^2}$$

giving the same results as before.

QUESTIONS FOR SOLUTION.

1141. (V. RAMASWAMI AIYAR):—If A, B, C, D be four concyclic points prove that the in- and ex-centres of the triangles ABC, BCD, CDA, DAB lie in a quartic which passes through A, B, C, D and touches the circle ABCD at the circular points at infinity.

If the equations of the circle and the rectangular hyperbola through A, B, C, D be $x^2 + y^2 - R^2 = 0$ and $xy + \lambda x + \mu y + \nu = 0$, show that the equation of the quartic is

$$\frac{x^2 + y^2 - R^2}{2R^2} = \frac{xy + \lambda x + \mu y + \nu}{xy - \nu}.$$

1142. (K. ANANDA RAO):—Let $\omega(n)$ and $\mu(n)$ be defined as follows:* $\omega(n) = 0$ if n has any prime factor repeated more than twice; $\omega(n) = (-1)^r 2^r$ otherwise, r being the number of non-repeated prime factors of n; $\mu(n) = 0$ if n has any repeated prime factors; $\mu(n) = (-1)^r$ otherwise, r having the same meaning as above.
Show that for $|x| < 1$

$$\sum_1^\infty \mu(n)\, x^n = \sum_1^\infty \frac{\omega(n)\, x^n}{1 - x^n}$$

1143. (K. ANANDA RAO):—If (i) $\lambda(n) = (-1)^s$, where s is the number of prime factors of n (each prime factor being counted according to its degree of multiplicity), (ii) the function $\phi(x)$ satisfies certain conditions, and (iii) $\psi(x) = \sum_1^\infty \lambda(n)\, \phi(nx)$, then show that $\phi(n) = \sum_q \psi(qx)$, the latter summation being taken over all the integers of q which have no repeated prime factors.

1144. (K. J. SANJANA, M.A.):—Show that

$$(a)\quad \frac{1}{20}\left\{\frac{20}{3^3 . 4^2} + \frac{47}{6^3 . 5^2} + \frac{84}{10^3 . 6^2} + \frac{131}{15^3 . 7^2} + \dots \textit{ad inf.}\right\} = \pi^2 - 9{\cdot}8\dot{6}.$$

(The value of the right-hand side is less than ·002938).

$$(b)\quad \frac{1}{25}\left\{\frac{19}{3^2 . 4^2} + \frac{46}{6^2 . 5^2} + \frac{79}{10^2 . 6^2} + \frac{118}{15^2 . 7^2} + \dots \textit{ad inf.}\right\} = 9{\cdot}87\dot{8} - \pi^2.$$

(The value of the right-hand side is less than ·009285).

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* 1 is not regarded as a prime number. The functions $\mu(n)$ and $\lambda(n)$ are well-known in the Analytic Theory of Numbers. See, for example, Landau, *Handuch der Lehre von der Verteilung des Primzahlen.*

1145. (N. DORAIRAJAN):—n rods $OA_1, OA_2, \ldots OA_n$ are hinged together at O, which is a plane joint. Show that the area of the plane polygon $A_1A_2 \ldots A_n$ is a maximum when the circles on A_1A_2, A_2A_3, ... A_nA_1 as diameters have a common orthogonal circle and that the perimeter is a maximum when all the sides teach a circle. When there are only 3 rods, can the triangle be constructed with a ruler and compasses?

1146. (A. NARASINGA RAO):—If three integers be chosen at random, the probability that no two of them have a common factor is

$$\frac{1}{\zeta(3)}\left\{1+\frac{3}{\zeta(2)}-\frac{3}{\zeta(3)}\right\}$$

where $\zeta(s)$ is the sum of the series $\sum_1^{\infty} n^{-s}$.

1147. (A. NARASINGA RAO):—(a_n) is a monotonic sequence of positive terms and a set S is constructed every member of which is the sum of *any* number of *distinct* a's. Show that if S is to be dense everywhere in some interval,

(1) zero must be the limit pt. of (a_n)

and (2) $a_n \not> a_{n+1} + a_{n+2} + \ldots$ to ∞ for sufficiently large values of n.

Conversely if (1) and (2) are satisfied by all the terms of the sequence, the set S is everywhere dense in the interval (0 to $\sum_1^{\infty} a_n$), if $\sum a_n$ be convergent, and in (0 to ∞) if $\sum a_n$ diverges.

1148. (A. A. KRISHNASWAMI AIYANGAR):—Given two vertices of a cyclic quadrilateral and the loci of the other two vertices, discuss the loci of the points of intersection of the three diagonals taken two by two. Give a purely geometrical investigation in the particular case when the given loci are concentric circles.

1149. (S. ANANTARAMA AIYAR):—P is a variable point in the plane of a $\triangle$ ABC which moves so that, if PD, PE, PF are drawn perpendicular to the sides, DEF remains constant in area. Prove geometrically that P moves on a circle.

1150. (MARTYN M. THOMAS, M.A.):—If ρ and ρ_n be the corresponding radii of curvature of the curve $p = f(r)$ and its nth negative pedal show that

$$\frac{nr^2-(n-1)\,p\,\rho_n}{(n+1)\,r^2-np\,\rho_n}=\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\cdots\frac{1}{2-}\,\frac{p\rho_n}{r^2}=\frac{r^{n-2}\rho}{p^{n-1}}.$$

to $n+1$ fractions.